MATRIX FUNCTIONS AND THEIR APPLICATION TO PROBLEMS OF HEAT AND MASS TRANSFER

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The boundary-value problem is solved for a system of differential equations of heat and mass transfer with unsteady sources of heat and mass to which the theory of matrix functions is applied.

The analytic theory of heat and mass transfer phenomena of bound matter in porous material is based on a system of differential equations of parabolic type [1]:

$$\left.\begin{aligned}\frac{\partial U_1}{\partial t} &= a_{11}\nabla^2 U_1 + a_{12}\nabla^2 U_2 + \theta_1(x, y, z, t)\\ \frac{\partial U_2}{\partial t} &= a_{21}\nabla^2 U_1 + a_{22}\nabla^2 U_2 + \theta_2(x, y, z, t)\end{aligned}\right\}. \tag{1}$$

The solution of system (1), obtained for various initial and boundary conditions, permits a more profound study of the mechanism of heat and mass transfer of bound matter. The development of effective new methods of solving (1) for various boundary conditions is of both theoretical and practical interest.

Let the generalized charge conductor be a semi-infinite three-dimensional medium with two degrees of freedom.

It is necessary to find functions $U_1(x, y, z, t)$, $U_2(x, y, z, t)$, satisfying (1) and the following boundary conditions of the second kind:

$$U_k(x, y, z, t)|_{t=0} = f_k(x, y, z), \tag{2}$$

$$\left.\frac{\partial U_k}{\partial x}\right|_{x=0} = \varphi_k(y, z, t) \tag{3}$$

$$(k = 1, 2)\ (0 \leqslant x < \infty, \ -\infty < y, z < \infty, \ t > 0).$$

We write

$$U = \begin{Vmatrix} U_1 \\ U_2 \end{Vmatrix}, \quad A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}, \quad \theta = \begin{Vmatrix} \theta_1 \\ \theta_2 \end{Vmatrix}, \tag{4}$$

then (1) in matrix form becomes

$$\frac{\partial U}{\partial t} = A\nabla^2 U + \theta(x, y, z, t). \tag{5}$$

The boundary condition for (5) will be:

$$U(x, y, z, t)|_{t=0} = f(x, y, z), \tag{6}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \varphi(y, z, t), \ (0 < x < \infty, \ -\infty < y, z < \infty, \ t > 0), \tag{7}$$

where $f = \begin{Vmatrix} f_1 \\ f_2 \end{Vmatrix}$, $\varphi = \begin{Vmatrix} \varphi_1 \\ \varphi_2 \end{Vmatrix}$ are single-column matrices.

We assume that U, f, θ, and φ are scalar functions and that the quantity A is a positive number.

In this case the solution of (5) with boundary conditions (6)-(7) is known to be:

$$U(x, y, z, t) = \frac{1}{(2\sqrt{\pi t})^3}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{1}{\sqrt{A^3}} \times \tag{8}$$

$$\exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4At}\right] \times$$

$$\left[1+\exp\left(-\frac{\alpha t}{At}\right)\right] f(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma -$$

$$-\frac{1}{4\sqrt{\pi^3}} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(t-\tau)^3}} \frac{1}{\sqrt{A}} \times$$

$$\times \exp\left[-\frac{x^2+(y-\beta)^2+(z-\gamma)^2}{4A(t-\tau)}\right] \varphi(\beta, \gamma, \tau)\, d\tau\, d\beta\, d\gamma + \tag{8}$$

(cont'd)

$$+\frac{1}{(2\sqrt{\pi})^3} \int_0^t \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(t-\tau)^3}} \frac{1}{\sqrt{A^3}} \times$$

$$\times \exp\left[-\frac{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}{4A(t-\tau)}\right]\left[1+\exp\left(-\frac{\alpha x}{A(t-\tau)}\right)\right] \times$$

$$\times\ \theta(\alpha, \beta, \gamma, \tau)\, d\tau\, d\alpha\, d\beta\, d\gamma.$$

It can be shown, by direct substitution of (8) into (5), that it is also a solution of the matrix differential equation.

On the left side of (8) we have a single column matrix. Therefore, each term on the right must also be a single column matrix. To determine the solution of the basic problem $U_1(x, y, z, t)$, $U_2(x, y, z, t)$, it is necessary to determine the elements of these matrices and to compare the matrix elements on both sides of (8). By summing the elements of the first rows, we obtain the solution $U_1(x, y, z, t)$, and by summing the second rows the function $U_2(x, y, z, t)$. Let the matrix

$$A^*(x, y, z, t; \alpha, \beta, \gamma, \tau) = \frac{1}{(\sqrt{A})^3} \times$$

$$\times \exp\left[-\frac{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}{4A(t-\tau)}\right]\left[1+\exp\left(-\frac{\alpha x}{A(t-\tau)}\right)\right] =$$

$$= \begin{Vmatrix} \overset{*}{a}_{11}(x, y, z, t; \alpha, \beta, \gamma, \tau) & \overset{*}{a}_{12}(x, y, z, t; \alpha, \beta, \gamma, \tau) \\ \overset{*}{a}_{21}(x, y, z, t; \alpha, \beta, \gamma, \tau) & \overset{*}{a}_{12}(x, y, z, t; \alpha, \beta, \gamma, \tau) \end{Vmatrix} = \| a^*_{ij} \|_1^2$$

and the matrix

$$A^{**}(x, y, z, t; 0, \beta, \gamma, \tau) = \left\| a^*_{ij}(x, y, z, t; 0, \beta, \gamma, \tau) \right\|_1^2 =$$

$$= \frac{1}{\sqrt{A}} \exp\left[-\frac{x^2+(y-\beta)^2+(z-\gamma)^2}{4A(t-\tau)}\right],$$

then

$$U(x, y, z, t) = \frac{1}{(2\sqrt{\pi t})^3} \times \tag{9}$$

$$\times \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A^*(x, y, z, t; \alpha, \beta, \gamma, 0) f(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma +$$

$$+\frac{1}{(2\sqrt{\pi})^3} \int_0^t \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(t-\tau)^3}} \times \tag{9}$$

$$\times A^*(x, y, z, t; \alpha, \beta, \gamma, \tau)\, \theta(\alpha, \beta, \gamma, \tau)\, d\tau\, d\alpha\, d\beta\, d\gamma -$$

$$-\frac{1}{4(\sqrt{\pi})^3} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(\sqrt{t-\tau})^3} \times$$

$$\times\ A^{**}(x, y, z, t; 0, \beta, \gamma, \tau)\, \varphi(\beta, \gamma, \tau)\, d\tau\, d\beta\, d\gamma.$$

Multiplying matrices $A^* \cdot f$, $A^* \cdot \theta$, A^{**} and comparing elements on both sides of (9), we obtain

$$U_k(x, y, z, t) = \frac{1}{(2\sqrt{\pi t})^3} \times$$
$$\times \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum_{j=1}^{2} f_j(\alpha, \beta, \gamma)\, a^*_{kj}(x, y, z, t; \alpha, \beta, \gamma, 0)\right] d\alpha\, d\beta\, d\gamma +$$

$$+ \frac{1}{(2\sqrt{\pi})^3}\int_0^t\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{(t-\tau)^3}} \times \sum_{j=1}^{2}\theta_j(\alpha, \beta, \gamma, \tau)\, a^*_{kj}(x, y, z, t; \alpha, \beta, \gamma, \tau)\right] d\tau\, d\alpha\, d\beta\, d\gamma - \quad (10)$$

$$- \frac{1}{4\sqrt{\pi^3}}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{(t-\tau)^3}} \times \left[\sum_{j=1}^{2}\varphi_j(\beta, \gamma, \tau)\, a^{**}_{kj}(x, y, z, t; 0, \beta, \gamma, \tau)\right] d\tau\, d\beta\, d\gamma,$$
$$(k = 1, 2).$$

Thus, the problem will be completely solved if we determine the functions $a^*_{kj}(x, y, z, t; \alpha, \beta, \gamma, \tau)$, $a^{**}_{kj}(x, y, z, t; 0, \beta, \gamma, \tau)$, i. e., if we determine the elements of matrices A^*, A^{**}. Consider an arbitrary function of the scalar argument $F(\lambda)$ and the corresponding function of matrix A, i. e., the matrix $F(A)$.

We set

$$|A - \lambda E| = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0, \quad (11)$$

and λ_1, λ_2 are roots of characteristic equation (11), i. e.,

$$\lambda_{1,2} = \frac{1}{2}(a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}, \quad \lambda_1 \neq \lambda_2 . \quad (12)$$

As in [2], the basic formula of the matrix function applicable to our case may be written in the form:

$$F(A) = F(\lambda_1) Z_1 + F(\lambda_2) Z_2 , \quad (13)$$

Matrices Z_1 and Z_2 are fully determined when A is given and do not depend on the choice of $F(\lambda)$. On the right of (13), the function $F(\lambda)$ is represented only by its values on the spectrum of A.

Matrices Z_1 and Z_2 are component matrices or components of matrix A and do not depend linearly on one another.

To determine Z_1 and Z_2, we put $F_1(\lambda) = \lambda - \lambda_1$, $F_2(\lambda) = \lambda - \lambda_2$, then

$$F_1(A) = \| A - \lambda_1 E \| = (\lambda_2 - \lambda_1) Z_2 , \; F_2(A) = \| A - \lambda_2 E \| = (\lambda_1 - \lambda_2) Z_1 , \quad (14)$$

whence

$$Z_1 = \frac{1}{\lambda_1 - \lambda_2}\begin{Vmatrix} a_{11}-\lambda_2 & a_{12} \\ a_{21} & a_{22}-\lambda_2 \end{Vmatrix}, \quad Z_2 = \frac{1}{\lambda_2 - \lambda_1}\begin{Vmatrix} a_{11}-\lambda_1 & a_{12} \\ a_{21} & a_{22}-\lambda_1 \end{Vmatrix}.$$

Substituting the values of Z_1 and Z_2 into (14), we obtain

$$F(A) = \frac{1}{\lambda_1 - \lambda_2}\left\{\begin{Vmatrix} a_{11}-\lambda_2 & a_{12} \\ a_{21} & a_{22}-\lambda_2 \end{Vmatrix} F(\lambda_1) - \begin{Vmatrix} a_{11}-\lambda_1 & a_{12} \\ a_{21} & a_{22}-\lambda_1 \end{Vmatrix} F(\lambda_2)\right\}. \quad (15)$$

Now we take

$$F(A) = A^*(x, y, z, t; \alpha, \beta, \gamma, \tau), \quad F(A) = A^{**}(x, y, z, t; 0, \beta, \gamma, \tau)$$

and substitute the values of these matrices on the right side of (9); then the first integral has the form

$$\frac{1}{(\lambda_1 - \lambda_2)(2\sqrt{\pi t})^3}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big\{\| A - \lambda_2 E \| \times \quad (16)$$

$$\times \frac{1}{\sqrt{\lambda_1^3}} \exp\left[-\frac{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}{4\lambda_1 t}\right]\left[1+\exp\left(-\frac{\alpha x}{\lambda_1 t}\right)\right]-$$

$$-\ \|A-\lambda_1 E\| \frac{1}{\sqrt{\lambda_2^3}} \exp\left[-\frac{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}{4\lambda_2 t}\right]\times \qquad (16)\ \text{(cont'd)}$$

$$\times \left[1+\exp\left(-\frac{\alpha x}{\lambda_2 t}\right)\right]\Bigg\} f(\alpha,\ \beta,\ \gamma)\, d\alpha\, d\beta\, d\gamma.$$

The two other integrals may be written similarly. Multiplying the matrices under the integral sign, and comparing matrix elements on both sides of (9), we obtain

$$U_j(x,\ y,\ z,\ t)=\frac{1}{(\lambda_1-\lambda_2)}\left\{\sum_{k=1}^{2}\frac{1}{(2\sqrt{\pi\lambda_k t})^3}\times\right.$$

$$\times \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f_{jk}^*(\alpha,\ \beta,\ \gamma) \exp\left[-\frac{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}{4\lambda_k t}\right]\times$$

$$\times \left[1+\exp\left(-\frac{\alpha x}{\lambda_k t}\right)\right] d\alpha\, d\beta\, d\gamma +$$

$$+\ \frac{1}{(2\sqrt{\pi})^3}\sum_{k=1}^{2}\frac{1}{\sqrt{\lambda_k^3}}\int_0^t \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\theta_{jk}^*(\alpha,\ \beta,\ \gamma,\ \tau)}{\sqrt{(t-\tau)^3}}\times \qquad (17)$$

$$\times\ \exp\left[-\frac{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}{4\lambda_k (t-\tau)}\right]\times$$

$$\times\ \left[1+\exp\left(-\frac{\alpha x}{\lambda_k (t-\tau)}\right)\right] d\tau\, d\alpha\, d\beta\, d\gamma -$$

$$-\ \frac{1}{4\sqrt{\pi^3}}\sum_{k=1}^{2}\frac{1}{\sqrt{\lambda_k}}\int_0^t \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\varphi_{jk}^*(\beta,\ \gamma,\ \tau)}{\sqrt{(t-\tau)^3}}\times$$

$$\left.\times\ \exp\left[-\frac{x^2+(y-\beta)^2+(z-\gamma)^2}{4\lambda_k (t-\tau)}\right] d\tau\, d\beta\, d\gamma\right\},$$

where

$$f_{11}^*=(a_{11}-\lambda_2) f_1 + a_{12} f_2,\quad f_{12}^* = -[(a_{11}-\lambda_1) f_1 + a_{12} f_2],$$
$$f_{21}^* = a_{21} f_1 + (a_{22}-\lambda_2) f_2,\quad f_{22}^* = -[a_{21} f_1 + (a_{22}-\lambda_1) f_2]. \qquad (18)$$

To determine θ_{kj}^*, φ_{kj}^* it is necessary to replace f_1 and f_2 in (18) by θ_1, θ_2 or φ_1, φ_2, respectively. When the spectra of the square matrix are multiple ($\lambda_1 = \lambda_2 = \lambda_0 > 0$), the solution is similar.

Equation (17) is the solution of (1) with boundary conditions (2)-(3). Similar solutions were obtained by the author in [4, 5] by the complex application of integral Fourier transforms with respect to the space coordinates and a Laplace transform with respect to time.

Finally, it should be noted that application of matrix functions to the system of differential equations of molecular transfer simplifies the solution of the boundary-value problem considerably and permits a number of new heat and mass transfer problems to be solved.

NOTATION

x, y, z – space coordinates; t – time; α, β, γ, τ – variables of integration; A – second-order square matrix; U, θ, f, φ – single column matrices; λ_1, λ_2 – eigenvalues (spectra) of matrix A; E – unit matrix.

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